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Fibre bundle formulation of nonrelativistic quantum mechanics: II. Equations of motion and observables

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Abstract

We propose a new systematic fibre bundle formulation of nonrelativistic quantum mechanics. The new form of the theory is equivalent to the usual one and is in harmony with the modern trends in theoretical physics and potentially admits new generalizations in different directions. In it the Hilbert space of a quantum system (from conventional quantum mechanics) is replaced by an appropriate Hilbert bundle of states and a pure state of the system is described by a lifting of paths or section along paths in this bundle. The evolution of a pure state is determined through the bundle (analogue of the) Schrödinger equation. Now the dynamical variables and density operators are described via liftings of paths or morphisms along paths in suitable bundles. The mentioned quantities are connected by a number of relations derived in this paper.

In the second part of this investigation are derived several forms of the bundle (analogue of the) Schrödinger equation governing the time evolution of state liftings of paths or sections along paths. We prove that up to a constant the matrix-bundle Hamiltonian, entering the bundle analogue of the matrix form of the conventional Schrödinger equation, coincides with the matrix of coefficients of the evolution transport. This allows us to interpret the Hamiltonian as a gauge field. We apply the bundle approach to the description of observables. It is shown that to any observable there corresponds a unique Hermitian lifting of paths or morphism along paths in corresponding bundles.

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1. Introduction

This paper is the second part of our investigation on a fibre bundle approach to nonrelativistic quantum mechanics. It is a straightforward continuation of [1].

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This paper is organized as follows.

Section 2 is devoted to the bundle analogues of the Schrödinger equation which are fully equivalent to it. In particular, in it is introduced the *matrix-bundle Hamiltonian*, which governs the quantum evolution through the *matrix-bundle Schrödinger equation*. The corresponding matrix of the evolution transport is found. It is proved that *up to a constant the matrix of the coefficients of evolution transport coincides with the matrix-bundle Hamiltonian*. On this basis is derived the *(invariant) bundle Schrödinger equation*. Geometrically this simply means that the corresponding state liftings are (parallelarly, or, more precisely, linearly) transported by means of the evolution transport along paths.

In section 3 the question of the bundle description of observables is considered. It turns out that *to any observable there corresponds a unique Hermitian lifting of paths* in the bundle of point-restricted morphisms over the base of the Hilbert bundle of states.

Section 4 concludes.

The notation of this paper is the same as that in [1] and we are not going to recall it here.

The references to sections, equations, footnotes etc from [1] are obtained from their sequential numbers in [1] by adding in front of them the Roman one (I) and a dot as a separator. For instance, section I.4 and (I.5.13) mean, respectively, section 4 and equation (5.13) (equation (13) in section 5) of [1].

Below, for reference purposes, we present a list of some essential equations of [1] which are used in this paper. Following the convention given above, we retain their original reference numbers.

$$\mathrm{i}\hbar \frac{\mathrm{d}\psi(t)}{\mathrm{d}t} = \mathcal{H}(t)\psi(t) \tag{I.2.6}$$

$$\mathrm{i}\hbar \frac{\partial\, \mathcal{U}(t,t_0)}{\partial t} = \mathcal{H}(t)\circ\, \mathcal{U}(t,t_0) \qquad \mathcal{U}(t_0,t_0) = \mathsf{id}_{\mathcal{F}} \tag{I.2.8}$$

$$\mathcal{H}(t) = \mathrm{i}\hbar \frac{\partial\, \mathcal{U}(t,t_0)}{\partial t}\circ\, \mathcal{U}^{-1}(t,t_0) = \mathrm{i}\hbar \frac{\partial\, \mathcal{U}(t,t_0)}{\partial t}\circ\, \mathcal{U}(t_0,t) \tag{I.2.9}$$

$$\langle\, \mathcal{A}\rangle^t_\psi := \langle\, \mathcal{A}(t)\rangle_{\psi(t)} := \langle\, \mathcal{A}(t)\rangle^t_\psi := \frac{\langle\psi(t)|\ \mathcal{A}(t)\psi(t)\rangle}{\langle\psi(t)|\psi(t)\rangle} \tag{I.2.11}$$

$$\Psi_\gamma(t) = l^{-1}_{\gamma(t)}(\psi(t)) \in F_{\gamma(t)} \tag{I.4.3}$$

$$\langle\cdot|\cdot\rangle_x = \langle l_x\cdot|l_x\cdot\rangle \qquad x\in M \tag{I.3.1}$$

$$\langle A^{\ddagger}\Phi_x|\Psi_x\rangle_x := \langle\Phi_x|A\Psi_x\rangle_x \qquad \Phi_x,\Psi_x\in F_x \tag{I.3.14}$$

$$\Psi_\gamma(t) = U_\gamma(t,s)\Psi_\gamma(s) \tag{I.5.7}$$

$$U_\gamma(t,s) = l^{-1}_{\gamma(t)}\circ\, \mathcal{U}(t,s)\circ l_{\gamma(s)} \qquad s,t\in J. \tag{I.5.10}$$

2. The bundle equations of motion

In conventional quantum mechanics, the time dependence of a state vector $\psi\in\mathcal{F}$ of a quantum system is governed via the Schrödinger equation (I.2.6). It is natural to expect the existence of an analogous equation for the state lifting Ψ replacing ψ by (I.4.3) in the bundle description of quantum mechanics. The derivation of this equation (or of its variants), which should only be in bundle terms, is the major purpose of the present section. Regardless of some technical problems, the idea is quite simple: using (I.4.3) and (I.5.7) or (I.5.10), one should transform the Schrödinger equation into 'pure' bundle terms. A realization of such a procedure is given below. The resulting (invariant) bundle equation of motion has an amazingly transparent geometrical meaning: it expresses the fact that the state liftings/sections are linearly transported along the reference path along which the quantum evolution is explored.

2.1. Derivation of the equations

If we substitute (I.5.11) into (I.2.6)–(I.2.10), we 'obtain' the 'bundle' analogues of (I.2.6)–(I.2.10). But they will be wrong! This is due to the fact that they will contain partial derivatives such as $\partial l_{\gamma(t)}/\partial t$, $\partial\Psi_\gamma(t)/\partial t$, and $\partial U_\gamma(t,t_0)/\partial t$, which are not defined at all. For instance, in the first case we must have $\partial l_{\gamma(t)}/\partial t = \lim_{\varepsilon\to 0}\left(\frac{1}{\varepsilon}(l_{\gamma(t+\varepsilon)} - l_{\gamma(t)})\right)$, but the 'difference' in this limit is not defined (for $\varepsilon \neq 0$) because $l_{\gamma(t+\varepsilon)}$ and $l_{\gamma(t)}$ act on different spaces, namely on $F_{\gamma(t+\varepsilon)}$ and $F_{\gamma(t)}$ respectively. The same is the situation with $\partial U_\gamma(t,t_0)/\partial t$. The most obvious is the contradiction in the relation $\partial\Psi_\gamma(t)/\partial t = \lim_{\varepsilon\to 0}\left(\frac{1}{\varepsilon}(\Psi_\gamma(t+\varepsilon) - \Psi_\gamma(t))\right)$, because $\Psi_\gamma(t+\varepsilon)$ and $\Psi_\gamma(t)$ belong to different (for $\varepsilon \neq 0$) vector spaces.

One can go through this difficulty by defining, for example, $\partial\Psi_\gamma(t)/\partial t$ like $l^{-1}_{\gamma(t)}\partial\psi_\gamma(t)/\partial t$ (cf (I.4.1)) but this does not lead to any important or new results.

To overcome this problem, we are going to introduce local bases (or coordinates) and to work with the matrices of the corresponding operators and vectors in them.

Let $\{e_a(x), a \in \Lambda\}$ be a basis in $F_x = \pi^{-1}(x)$, $x \in M$. The indices $a, b, c, \ldots \in \Lambda$ may take discrete, or continuous, or both, values. More precisely, the set Λ has a decomposition $\Lambda = \Lambda_d \bigcup \Lambda_c$, where Λ_d is a union of (finite or countable) subsets of $\mathbb{N}$ (or, equivalently, of $\mathbb{Z}$) and Λ_c is union of subsets of $\mathbb{R}$ (or of $\mathbb{C}$). Note that Λ_d or Λ_c, but not both, can be empty. This is why sums such as[2] $\lambda^a e_a(x)$ or $\lambda^a\mu_a$ for $a \in \Lambda$ and $\lambda^a, \mu_a \in \mathbb{C}$ must be understood as a sum over the discrete (enumerable) part(s) of Λ, if any, plus the (Stieltjes or Lebesgue) integrals over the continuous part(s) of Λ, if any. For instance $\lambda^a e_a(x) := \sum_{a\in\Lambda}\lambda^a e_a(x) := \sum_{a\in\Lambda_d}\lambda^a e_a(x) + \int_{a\in\Lambda_c}\lambda^a e_a(x)\,\mathrm{d}a$. For this reason it is better to write $\sum\!\!\!\!\int_{a\in\Lambda} := \sum_{a\in\Lambda_d} + \int_{a\in\Lambda_c}\mathrm{d}a$ instead of $\sum_{a\in\Lambda}$, but we shall avoid this complicated notation by using the assumed summation convention on indices repeated on different levels[3].

The matrices corresponding to vectors or operators in a given field of bases will be denoted with the same symbol but in **boldface**, for example[4] $\boldsymbol{U}_\gamma(t,s) := \left[\left(U_\gamma(t,s)\right)^a_{\ b}\right]$ and $\boldsymbol{\Psi}_\gamma(s) := \left[\Psi^a_\gamma(s)\right]$, where $U_\gamma(t,s)\,(e_b(\gamma(s))) =: \left(U_\gamma(t,s)\right)^a_{\ b}\, e_a(\gamma(t))$ and $\Psi_\gamma(s) =: \Psi^a_\gamma(s)e_a(\gamma(s))$.

Analogously, we suppose that in $\mathcal{F}$ there will be a fixed basis $\{f_a(t),\ a \in \Lambda\}$ with respect to which we shall use the same bold-faced matrix notation, for instance $\boldsymbol{\mathcal{U}}(t,s) = \left[\,\mathcal{U}^b_{\ a}(t,s)\right]$, $\mathcal{U}(t,s)\,(f_a(s)) =: (\,\mathcal{U}(t,s))^b_{\ a}\, f_b(t)$, $\boldsymbol{\psi}(t) = [\psi^a(t)]$, $\psi(t) =: \psi^a(t)f_a(t)$ and, finally, $\boldsymbol{l}_x(t) = \left[\,(l_x)^b_{\ a}(t)\right]$, $l_x\,(e_a(x)) =: (l_x)^b_{\ a}(t)f_b(t)$. Generally, $\boldsymbol{l}_x(t)$ depends on x and t, but if $x = \gamma(s)$ for some $s \in J$, we put $t = s$ as from physical reasons is clear that $F_{\gamma(t)}$ corresponds to $\mathcal{F}$ at the 'moment' t, i.e. the components of $l_{\gamma(s)}$ are with respect to $\{e_a(\gamma(s))\}$ and $\{f_a(s)\}$. The same remark concerns 'two-point' objects such as $U_\gamma(t,s)$ and $\mathcal{U}(t,s)$ whose components will be taken with respect to pairs of bases such as $(\{e_a(\gamma(t))\}, \{e_a(\gamma(s))\})$ and $(\{f_a(t)\}, \{f_a(s)\})$ respectively.

Evidently, the equations (I.4.1) and (I.5.7)–(I.5.10) remain valid *mutatis mutandis* in the introduced matrix notation: the kernel letters have to be made bold faced, the operator composition (product) must be replaced by matrix multiplication and the identity map id_{F_x} has to be replaced by the unit matrix $\mathbb{1}_{F_x} := \left[\delta^b_a\right] := \left[\left(\mathsf{id}_{F_x}\right)^b_{\ a}\right]$ of F_x in $\{e_a(x)\}$. Here $\delta^b_a = 1$ for $a = b$ and $\delta^b_a = 0$ for $a \neq b$, which means that $e_a(x) = \delta^b_a e_b(x)$. For instance, using the above definitions, one verifies that (I.5.10) is equivalent to

$$\boldsymbol{U}_\gamma(t,s) = \boldsymbol{l}^{-1}_{\gamma(t)}(t)\,\boldsymbol{\mathcal{U}}(t,s)\boldsymbol{l}_{\gamma(s)}(s). \tag{2.1}$$

[2] Here and henceforth in this paper, we use the Einstein rule for summation over indices repeated on different levels.
[3] For details concerning infinite-dimensional matrices see, for instance, [2] and [3, chapter 7, section 18]. A comprehensive presentation of the theory of infinite matrices is given in [4]; this book is mainly devoted to infinite discrete matrices but it also contains some results on continuous infinite matrices related to Hilbert spaces.
[4] The matrices $\boldsymbol{\mathcal{U}}(t,s)$ and $\boldsymbol{U}(t,s)$ are closely related to propagator functions [5], but we will not need these explicit connections. For explicit calculations and construction of $\boldsymbol{\mathcal{U}}(t,s)$, see [5, sections 21, 22].

Let $\boldsymbol{\Omega}(x) := \left[\Omega_a^{\ b}(x)\right]$ and $\boldsymbol{\omega}(t) := \left[\omega_a^{\ b}(t)\right]$ be nondegenerate matrices. The changes

$$\{e_a(x)\} \to \{e'_a(x) := \Omega_a^{\ b}(x) e_b(x)\} \qquad \{f_a(t)\} \to \{e'_a(t) := \omega_a^{\ b}(t) e_b(t)\}$$

of the bases in F_x and $\mathcal{F}$, respectively, lead to the transformation of the matrices of the components of $\Phi_x \in F_x$ and $\phi \in \mathcal{F}$ according to

$$\boldsymbol{\Phi}_x \mapsto \boldsymbol{\Phi}'_x = (\boldsymbol{\Omega}^\top(x))^{-1}\boldsymbol{\Phi}_x \qquad \boldsymbol{\phi} \mapsto \boldsymbol{\phi}' = (\boldsymbol{\omega}^\top(t))^{-1}\boldsymbol{\phi}. \tag{2.2}$$

Here the superscript $\top$ means matrix transposition, for example $\boldsymbol{\Omega}^\top(x) := \left[\left(\Omega^\top(x)\right)^a_{\ b}\right]$ with $\left(\Omega^\top(x)\right)^a_{\ b} := \Omega_b^{\ a}(x)$. One easily verifies the transformation

$$\boldsymbol{l}_x(t) \mapsto \boldsymbol{l}'_x(t) = \left(\boldsymbol{\omega}^\top(t)\right)^{-1}\boldsymbol{l}_x(t)\boldsymbol{\Omega}^\top(x) \tag{2.3}$$

of the components of the linear isomorphisms $l_x : F_x \to \mathcal{F}$ under the above changes.

For any operator $\mathcal{A}(t) : \mathcal{F} \to \mathcal{F}$ we have

$$\boldsymbol{\mathcal{A}}(t) \mapsto \boldsymbol{\mathcal{A}}'(t) = (\boldsymbol{\omega}^\top(t))^{-1}\boldsymbol{\mathcal{A}}(t)\boldsymbol{\omega}^\top(t). \tag{2.4}$$

Analogously, if $A(t)$ is a morphism of (F, π, M), i.e. if $A : F \to F$ and $\pi \circ A = \mathsf{id}_M$, and $A_x := A(t)|_{F_x}$, then

$$\boldsymbol{A}_x(t) \mapsto \boldsymbol{A}'_x(t) = \left(\boldsymbol{\Omega}^\top(t)\right)^{-1}\boldsymbol{A}_x(t)\boldsymbol{\Omega}^\top(t). \tag{2.5}$$

Note that the components of $\mathcal{U}(t, s)$, when referred to a pair of bases $\{e_a(t)\}$ and $\{e_a(s)\}$, transform according to

$$\boldsymbol{\mathcal{U}}(t, s) \mapsto \boldsymbol{\mathcal{U}}'(t, s) = \left(\boldsymbol{\omega}^\top(t)\right)^{-1}\boldsymbol{\mathcal{U}}(t, s)\boldsymbol{\omega}^\top(s). \tag{2.6}$$

Analogously, the change $\{e_a(\gamma(t))\} \to \{e'_a(t;\gamma) := \Omega_a^{\ b}(t;\gamma)e_b(\gamma(t))\}$, with a nondegenerate matrix $\boldsymbol{\Omega}(t;\gamma) := \left[\Omega_a^{\ b}(t;\gamma)\right]$ along γ, implies[5]

$$\boldsymbol{U}_\gamma(t, s) \mapsto \boldsymbol{U}'_\gamma(t, s) = \left(\boldsymbol{\Omega}^\top(t;\gamma)\right)^{-1}\boldsymbol{U}_\gamma(t, s)\boldsymbol{\Omega}^\top(s;\gamma). \tag{2.7}$$

Substituting $\psi(t) = \psi^a(t) f_a(t)$ into (I.2.6), we obtain the *matrix Schrödinger equation*

$$\frac{\mathrm{d}\boldsymbol{\psi}(t)}{\mathrm{d}t} = \boldsymbol{\mathcal{H}}^m(t)\boldsymbol{\psi}(t) \tag{2.8}$$

where

$$\boldsymbol{\mathcal{H}}^m(t) := \boldsymbol{\mathcal{H}}(t) - \mathrm{i}\hbar\boldsymbol{E}(t) \tag{2.9}$$

is the *matrix Hamiltonian* (in the Hilbert space description). Here $\boldsymbol{E}(t) = \left[E_a^{\ b}(t)\right]$ determines the expansion of $\mathrm{d}f_a(t)/\mathrm{d}t$ over $\{f_a(t)\} \subset \mathcal{F}$, that is $\mathrm{d}f_a(t)/\mathrm{d}t = E_a^{\ b}(t) f_b(t)$; if $f_a(t)$ are independent of t, which is the usual case, we have $\boldsymbol{E}(t) = 0$. In the last case $\boldsymbol{\mathcal{H}}^m = \boldsymbol{\mathcal{H}}$. It is important to note that $\boldsymbol{\mathcal{H}}^m$ is independent of $\boldsymbol{E}(t)$. In fact, applying (I.2.9) to the basic vector $f_a(t)$, we obtain $\mathcal{H}(t) f_a(t) = \mathrm{i}\hbar[(\frac{\partial}{\partial t}\,\mathcal{U}(t, t_0)) f_b(t_0)]\,\mathcal{U}_a^{\ b}(t_0, t) = \mathrm{i}\hbar[\frac{\partial}{\partial t}(f_c(t)\,\mathcal{U}_b^{\ c}(t, t_0))]\,\mathcal{U}_a^{\ b}(t_0, t)$, so

$$\boldsymbol{\mathcal{H}}(t) = \mathrm{i}\hbar\frac{\partial\boldsymbol{\mathcal{U}}(t, t_0)}{\partial t}\boldsymbol{\mathcal{U}}(t_0, t) + \mathrm{i}\hbar\boldsymbol{E}(t) \tag{2.10}$$

which leads to

$$\boldsymbol{\mathcal{H}}^m(t) = \mathrm{i}\hbar\frac{\partial\boldsymbol{\mathcal{U}}(t, t_0)}{\partial t}\boldsymbol{\mathcal{U}}(t_0, t). \tag{2.11}$$

[5] Cf [6, equation (2.11)] or [7, equation (4.10)], where the notation $\boldsymbol{L}(t, s;\gamma) = H(t, s;\gamma) = \boldsymbol{U}_\gamma(s, t;\gamma)$ and $A(t) = \boldsymbol{\Omega}^\top(t;\gamma)$ is used.

Substituting the matrix form of (I.4.1) into (2.8), we find the *matrix-bundle Schrödinger equation*

$$\mathrm{i}\hbar \frac{\mathrm{d}\boldsymbol{\Psi}_\gamma(t)}{\mathrm{d}t} = \boldsymbol{H}_\gamma^m(t)\boldsymbol{\Psi}_\gamma(t) \tag{2.12}$$

where the *matrix-bundle Hamiltonian* is

$$\boldsymbol{H}_\gamma^m(t) := \boldsymbol{l}_{\gamma(t)}^{-1}(t)\boldsymbol{\mathcal{H}}(t)\boldsymbol{l}_{\gamma(t)}(t) - \mathrm{i}\hbar \boldsymbol{l}_{\gamma(t)}^{-1}(t)\left(\frac{\mathrm{d}\boldsymbol{l}_{\gamma(t)}(t)}{\mathrm{d}t} + \boldsymbol{E}(t)\boldsymbol{l}_{\gamma(t)}(t)\right). \tag{2.13}$$

Combining (2.9) and (2.13), we find the following connection between the conventional and matrix-bundle Hamiltonians:

$$\boldsymbol{H}_\gamma^m(t) = \boldsymbol{l}_{\gamma(t)}^{-1}(t)\boldsymbol{\mathcal{H}}^m(t)\boldsymbol{l}_{\gamma(t)}(t) - \mathrm{i}\hbar \boldsymbol{l}_{\gamma(t)}^{-1}(t)\frac{\mathrm{d}\boldsymbol{l}_{\gamma(t)}(t)}{\mathrm{d}t}. \tag{2.14}$$

Remark 2.1. Choosing $e_a(x) = l_x^{-1}(f_a)$ for $\mathrm{d}f_a(t)/\mathrm{d}t = 0$, we obtain $\boldsymbol{l}_x(t) = [\delta_a^b]$. Then $\boldsymbol{H}_\gamma(t) = \boldsymbol{\mathcal{H}}(t)$. So, as $\mathcal{H}^\dagger = \mathcal{H}$, we have $(\boldsymbol{H}_\gamma^m(t))^\dagger = \boldsymbol{\mathcal{H}}^\dagger(t) = \boldsymbol{\mathcal{H}}(t) = \boldsymbol{H}_\gamma^m(t)$, where we also use the dagger (†) to denote matrix Hermitian conjugation. Here $\boldsymbol{H}_\gamma^m(t)$ is a Hermitian matrix in the chosen basis, but in other bases it may not be as such (see (2.24) below). Analogously, choosing $\{f_a(t)\}$ such that $\boldsymbol{E}(t) = 0$, we see that $\boldsymbol{\mathcal{H}}^m(t) = \boldsymbol{\mathcal{H}}(t)$ is a Hermitian matrix, otherwise it may not be as such.

Remark 2.2. Note that, due to (2.14), the transition $\boldsymbol{\mathcal{H}}^m \to \boldsymbol{H}_\gamma^m$ is very much like a gauge (or connection) transformation [8] (see also below (2.22)–(2.24)).

Because of (2.12) and (I.5.7) there is a bijective correspondence between $\boldsymbol{U}_\gamma$ and $\boldsymbol{H}_\gamma^m$ expressed through the initial-value problem (cf (I.2.8))

$$\mathrm{i}\hbar \frac{\partial \boldsymbol{U}_\gamma(t,t_0)}{\partial t} = \boldsymbol{H}_\gamma^m(t)\boldsymbol{U}_\gamma(t,t_0) \qquad \boldsymbol{U}_\gamma(t_0,t_0) = \mathbb{1}_{F_{\gamma(t_0)}} \tag{2.15}$$

or via the integral equation equivalent to it

$$\boldsymbol{U}_\gamma(t,t_0) = \mathbb{1}_{F_{\gamma(t_0)}} + \frac{1}{\mathrm{i}\hbar}\int_{t_0}^{t} \boldsymbol{H}_\gamma^m(\tau)\boldsymbol{U}_\gamma(\tau,t_0)\,\mathrm{d}\tau. \tag{2.16}$$

So, if $\boldsymbol{H}_\gamma^m$ is given, we have (cf (I.2.10))

$$\boldsymbol{U}_\gamma(t,t_0) = \mathrm{Texp}\int_{t_0}^{t} \frac{1}{\mathrm{i}\hbar}\boldsymbol{H}_\gamma^m(\tau)\,\mathrm{d}\tau \tag{2.17}$$

and, conversely, if $\boldsymbol{U}_\gamma$ is given, then (cf (I.2.9) and (2.11))[6]

$$\boldsymbol{H}_\gamma^m(t) = \mathrm{i}\hbar \frac{\partial \boldsymbol{U}_\gamma(t,t_0)}{\partial t}\boldsymbol{U}_\gamma^{-1}(t,t_0) = \mathrm{i}\hbar \frac{\partial \boldsymbol{U}_\gamma(t,t_0)}{\partial t}\boldsymbol{U}_\gamma(t_0,t). \tag{2.18}$$

The next step is to write the above matrix equations into an invariant, i.e. basis-independent, form. For this purpose we shall use the derivation introduced in [6, 7] along paths uniquely corresponding to any linear transport along paths in a vector bundle.

According to definitions I.3.3 and I.3.4 the derivation along paths corresponding to the bundle evolution transport U is a linear mapping

$$D : \mathrm{PLift}^1(F,\pi,M) \to \mathrm{PLift}^0(F,\pi,M)$$

[6] Expressions like $(\partial\, \mathcal{U}(t,t_0)/\partial t)\, \mathcal{U}(t_0,t)$, $(\partial \boldsymbol{U}_\gamma(t,t_0)/\partial t)\boldsymbol{U}_\gamma^{-1}(t,t_0)$, and $\mathcal{U}(t,t_0)\, \mathcal{U}(t_0,t_1)$ are independent of t_0 due to [6, propositions 2.1 and 2.4] or [7, propositions 2.1 and 2.4] (see also (I.3.23), (I.3.44), and [9, lemma 3.1]).

$\mathrm{PLift}^k(F, \pi, M)$ being the set of C^k liftings of paths from M to F, such that for every C^1 lifting λ of paths and every path $\gamma : J \to M$ we have $D : \lambda \mapsto D(\lambda) = D\lambda$ and $D\lambda : \gamma \mapsto D^\gamma(\lambda) = (D\lambda)_\gamma$ is defined by $D^\gamma\lambda : s \mapsto D^\gamma_s\lambda \in F_{\gamma(s)}$ with

$$D^\gamma_s(\lambda) := \lim_{\varepsilon\to 0}\left\{\frac{1}{\varepsilon}\left[U_\gamma(s, s+\varepsilon)\lambda_\gamma(s+\varepsilon) - \lambda_\gamma(s)\right]\right\} \tag{2.19}$$

where $\lambda : \gamma \mapsto \lambda_\gamma$.

By (I.3.42) (see also [6, equation (2.27)] or [7, proposition 4.2]) the explicit local form of (2.19) in a frame $\{e_i(\cdot, \gamma)\}$ along γ is

$$D^\gamma_s\lambda = \left(\frac{\mathrm{d}\lambda^a_\gamma(s)}{\mathrm{d}s} + \Gamma^a_{\ b}(s;\gamma)\lambda^b_\gamma(s)\right)e_a(s;\gamma) \tag{2.20}$$

where the *coefficients* $\Gamma^b_{\ a}(s;\gamma)$ of U are defined by (cf (I.3.43))

$$\Gamma^b_{\ a}(s;\gamma) := \left.\frac{\partial\left(U_\gamma(s,t)\right)^b_{\ a}}{\partial t}\right|_{t=s} = -\left.\frac{\partial\left(U_\gamma(t,s)\right)^b_{\ a}}{\partial t}\right|_{t=s}. \tag{2.21}$$

Using (I.5.9) and (2.18), both for $t_0 = t$, we see that

$$\boldsymbol{\Gamma}_\gamma(t) := [\Gamma^b_{\ a}(t;\gamma)] = -\frac{1}{\mathrm{i}\hbar}\boldsymbol{H}^m_\gamma(t) \tag{2.22}$$

which expresses a fundamental result: *up to a constant (equal to $-\mathrm{i}\hbar$) the matrix-bundle Hamiltonian coincides with the matrix of coefficients of the bundle evolution transport* (in a given field of bases). Let us recall that, using other arguments, an analogous result was obtained in [10, section 5].

There are two invariant operators corresponding to the Hamiltonian $\mathcal{H}$ in $\mathcal{F}$: the evolution transport U and the corresponding derivation along paths D. Equations (2.12)–(2.22), as well as the general results of [6, section 2] and [7, section 4], imply that these three operators, namely $\mathcal{H}$, U and D, are equivalent in a sense that if one of them is given, then the remaining ones are uniquely determined.

Example 2.1. *Let $\{e_a(x)\}$ be fixed by $e_a(x) = l_x^{-1}(f_a)$ for $\mathrm{d}f(t)/\mathrm{d}t = 0$. Then $\boldsymbol{H}^m_\gamma(t)$ is a Hermitian matrix (see remark 2.1). Consequently, in this case, $\boldsymbol{\Gamma}_\gamma(t)$ is anti-Hermitian, i.e. $\left(\boldsymbol{\Gamma}_\gamma(t)\right)^\dagger = -\boldsymbol{\Gamma}_\gamma(t)$. Note that for other choices of the bases this property may not hold.*

Example 2.2. *Let $\mathcal{H}$ be given and independent of t, i.e. $\partial\mathcal{H}(t)/\partial t = 0$, and $\{e_a(x)\}$ be fixed by $e_a(x) = l_x^{-1}(f_a)$ for $\mathrm{d}f(t)/\mathrm{d}t = 0$. Then $\boldsymbol{l}_x(t) = \left[\delta^b_a\right]$ with δ^b_a defined above. Equations (2.13) and (2.22) yield $\boldsymbol{H}^m_\gamma(t) = \boldsymbol{\mathcal{H}}(t)$ and $\boldsymbol{\Gamma}_\gamma(t) = -\boldsymbol{\mathcal{H}}(t)/\mathrm{i}\hbar$. Finally, now the solution of (2.15) is $\boldsymbol{U}_\gamma(t, t_0) = \exp\left(\boldsymbol{\mathcal{H}}(t)(t - t_0)/\mathrm{i}\hbar\right)$ (cf (2.17)).*

According to [6, equation (2.30)] (or [7, equation (4.11)]) and footnote I.31, if a basis $\{e_a(t;\gamma)\}$ is changed to $\{e'_a(t;\gamma) = \Omega_a^{\ b}(t;\gamma)e_b(\gamma(t))\}$ with $\det\boldsymbol{\Omega}(t;\gamma) \neq 0$, $\boldsymbol{\Omega}(t;\gamma) := \left[\Omega_a^{\ b}(t;\gamma)\right]$, then $\boldsymbol{\Gamma}_\gamma(t)$ transforms into[7] (see (I.3.46))

$$\boldsymbol{\Gamma}'_\gamma(t) = (\boldsymbol{\Omega}^\top(t;\gamma))^{-1}\boldsymbol{\Gamma}_\gamma(t)\boldsymbol{\Omega}^\top(t;\gamma) + (\boldsymbol{\Omega}^\top(t;\gamma))^{-1}\frac{\mathrm{d}\boldsymbol{\Omega}^\top(t;\gamma)}{\mathrm{d}t}. \tag{2.23}$$

This result is also a corollary of (2.6) and (2.21).

Hence (see (2.22)), the matrix-bundle Hamiltonian undergoes the change $\boldsymbol{H}^m_\gamma(t) \mapsto {}'\boldsymbol{H}^m_\gamma(t)$ where

$${}'\boldsymbol{H}^m_\gamma(t) = (\boldsymbol{\Omega}^\top(t;\gamma))^{-1}\boldsymbol{H}^m_\gamma(t)\boldsymbol{\Omega}^\top(t;\gamma) - \mathrm{i}\hbar(\boldsymbol{\Omega}^\top(t;\gamma))^{-1}\frac{\mathrm{d}\boldsymbol{\Omega}^\top(t;\gamma)}{\mathrm{d}t}. \tag{2.24}$$

[7] In [6,7] the matrix $A(t) = \boldsymbol{\Omega}^\top(t;\gamma)$ is used instead of $\boldsymbol{\Omega}(t;\gamma)$.

This result can also be deduced from (2.14).

Now we are able to write the matrix-bundle Schrödinger equation (2.12) in an invariant form. Substituting (2.22) into (2.12) and using (2.20), we find that (2.12) is equivalent to

$$D_t^{\gamma}\Psi = 0 \tag{2.25}$$

or, as $t \in J$ is arbitrary, to

$$D^{\gamma}\Psi = 0. \tag{2.26}$$

Since $\gamma : J \to M$ is arbitrary, the last equation can be rewritten as

$$D\Psi = 0. \tag{2.27}$$

This is the (invariant) *bundle Schrödinger equation* (for the state liftings). Since it coincides with the *linear transport equation* [11, definition 5.2] for the evolution transport, it has a very simple and fundamental geometrical meaning. By [11, proposition 5.4] this is equivalent to the statement that Ψ_γ *is a lifting (linearly) transported along* γ with respect to the evolution transport (expressed in other terms via (I.5.7); see [9, definition 2.2]). Note that (2.25) and (I.5.7) are compatible as [7, equation (4.5)] is fulfilled (see also [6, equation (2.25)]): $D_t^{\gamma}(\overline{U}) \equiv 0,\ t \in J$ where $\overline{U} \in \mathrm{PLift}(F, \pi, M)$ is the lifting of paths generated by U (see definition I.3.5). Moreover, if D is given (independently of U, e.g. through (2.20)), from [11, proposition 5.4] it follows that U is the unique solution of the (invariant) initial-value problem[8]

$$D_t^{\gamma}(\overline{U}) = 0 \qquad \overline{U}_{\gamma}(t_0, t_0) = \mathsf{id}_{F_{\gamma(t_0)}} \tag{2.28}$$

for fixed $t_0 \in J$. Since here $t \in J$ and $\gamma : J \to M$ are arbitrary, the equation in this initial-value problem is equivalent to

$$D^{\gamma}(\overline{U}) = 0 \tag{2.29}$$

or to

$$D(\overline{U}) = 0. \tag{2.30}$$

This is the *bundle Schrödinger equation for the evolution transport* U.

Remark 2.3. Mathematically, equation (2.27) (or (2.25)) is a trivial corollary of (I.5.7) and (I.3.40). But this derivation of (2.27) leaves open the problem for its relation (equivalence) with the Schrödinger one. Besides, such a 'quick' derivation of (2.27) leaves hidden the above-mentioned properties of the matrix Hamiltonians, in particular the fundamental relation (2.22).

2.2. Inferences

Thus we see that there are two equivalent ways to describe the unitary evolution of a quantum system: (i) by means of the evolution operator $\mathcal{U}$ (see (I.2.1)) or by the Hermitian Hamiltonian $\mathcal{H}$ (see (I.2.6)) in the Hilbert space $\mathcal{F}$ (which is the typical fibre in the bundle description) and (ii) via the evolution transport U (see (I.5.7)), which is a Hermitian (and unitary) transport along paths, or the derivation D along paths (see (2.25)) in the Hilbert bundle (F, π, M). In the bundle description U corresponds to $\mathcal{U}$ (see (I.5.10)) and D to $\mathcal{H}$ (see (2.20) and (2.22)).

We derived the bundle Schrödinger equation (2.27) from the 'classical' Schrödinger equation (I.2.6); the equivalence of the two equations is evident from the above considerations.

Now we have at our disposal all tools required for pure bundle description of the evolution of a (pure) quantum system.

[8] In fact, (2.28) is the inversion of (2.19) with respect to U.

Given a system characterized by a derivation D along paths, if the system's bundle state victor Ψ^0_γ is known along $\gamma : J \to M$ at a point $t_0 \in J$, the state lifting Ψ of paths is a solution of the bundle Schrödinger equation (2.25) under the initial condition

$$\Psi_\gamma(t_0) = \Psi^0_\gamma. \tag{2.31}$$

By virtue of (2.20), equation (2.25) and condition (2.31) form a standard initial-value problem for a first-order system of ordinary differential equations (with respect to the time t) which has solutions along γ [12][9]. This solution is

$$\Psi_\gamma(t) = U_\gamma(t, t_0)\Psi^0_\gamma$$

where the evolution transport U could be found as the unique solution of the initial-value problem (2.28)

Above we supposed the system to be described via a derivation D along paths instead of by a Hamiltonian $\mathcal{H}$. These are equivalent approaches. Actually, in a local field of bases along γ, the matrix of $\mathcal{H}$ and that of the coefficients of D are connected by (2.22) and (2.13) and, hence, can uniquely be expressed through each other. Consequently, if the Hamiltonian $\mathcal{H}$ is known, one can construct from it the derivation D and *vice versa*. In the next section we shall see that to the Hamiltonian $\mathcal{H}$, as an observable, in the bundle description corresponds, besides D, a suitable lifting H of paths or (multiple-valued) sections along paths, the bundle Hamiltonian.

Now we shall derive a new form of the bundle Schrödinger equation in terms of the derivation $\tilde{D}$ along paths in $\mathrm{mor}_M(F, \pi, M)$ induced by the derivation D along paths generated by the evolution transport[10] U.

Applying equation (2.20), we can find the explicit (matrix of the) action of $\tilde{D}^\gamma_t(C) := D^\gamma_t \circ C$, $C \in \mathrm{PLift}^1(\mathrm{mor}_M(F, \pi, M))$, on a state lifting Ψ provided the lift C_γ is *linear*.

Let $[X]$ be the matrix of a vector or an operator X in $\{e_a\}$. Due to (2.20), we have

$$[(\tilde{D}^\gamma_t(C))\Psi] = \left(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{C}_\gamma(t)\right)\boldsymbol{\Psi}_\gamma(t) + \boldsymbol{C}_\gamma(t)\left(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\Psi}_\gamma(t)\right) + \boldsymbol{\Gamma}_\gamma(t)\boldsymbol{C}_\gamma(t)\boldsymbol{\Psi}_\gamma(t)$$

which is a special case of (I.3.50). Substituting here $\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\Psi}_\gamma(t)$ from (2.12) and using (2.22), we obtain the matrix equation

$$[(\tilde{D}^\gamma_t(C))\Psi] = \left(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{C}_\gamma(t)\right)\boldsymbol{\Psi}_\gamma(t) + \left[\boldsymbol{\Gamma}_\gamma(t), \boldsymbol{C}_\gamma(t)\right]_- \boldsymbol{\Psi}_\gamma(t) \tag{2.32}$$

where $[\cdot, \cdot]_-$ denotes the commutator of matrices, or

$$[\tilde{D}^\gamma_t(C)] = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{C}_t + \left[\boldsymbol{\Gamma}_\gamma(t), \boldsymbol{C}_\gamma(t)\right]_- . \tag{2.33}$$

Comparing this equation with (I.3.49), we obtain $[\tilde{D}^\gamma_t(C)] = [{}^\circ D^\gamma_t(C)]$ where ${}^\circ D$ is the derivation along paths in $\mathrm{mor}_M(F, \pi, M)$ associated with D according to (I.3.48). Therefore the invariant bundle form of (2.33) is

$$\tilde{D}(C) = {}^\circ D(C) \tag{2.34}$$

where $C \in \mathrm{PLift}^1(\mathrm{mor}_M(F, \pi, M))$ acts *only on state liftings* according to (I.3.34) and C_γ is *linear*. Equivalently, we can write

$$\tilde{D}|_{\mathcal{O}} = {}^\circ D|_{\mathcal{O}} \tag{2.35}$$

with $\mathcal{O}$ being the set of just-described liftings C.

[9] This initial-value problem is analogous (and equivalent) to that for the Schrödinger equation (I.2.6) and condition (I.2.7).

[10] For the notation and corresponding definitions, see section I.3.3, in particular, equations (I.3.33)–(I.3.37).

We derived (2.34) under the assumption that C_γ is linear and C acts on state liftings, i.e. on ones satisfying the matrix-bundle Schrödinger equation (2.12). Conversely, if we apply (2.33) to some vector $\Phi_\gamma(t) \in F_{\gamma(t)}$ and compare the result with that for $\big(D_t^\gamma(C)\big)(\Phi)$ obtained through (2.20) (see above), we see that $\Phi_\gamma(t)$ satisfies (2.12). Consequently, equation (2.34) with linear C_γ is valid if and only if C is applied on liftings representing the evolution of a quantum system. Hence Ψ is a state lifting, i.e. it satisfies, for instance, the bundle Schrödinger equation (2.27), iff the equation

$$(\tilde{D}(C))\Psi = (^\circ D(C))\Psi \tag{2.36}$$

is valid for every lifting C in the bundle of restricted morphisms such that C_γ is linear for every γ. In particular, (2.36) is valid for the (Hermitian) liftings (of paths) corresponding to observables (see further section 3) and Ψ satisfying the bundle Schrödinger equation (2.27).

The above overall discussion shows the equivalence of (2.36) (for every C with C_γ linear) to the Schrödinger equation (in any one of its (equivalent) forms mentioned until now). That is why (2.36) can be called the *matrix-lifting Schrödinger equation.*

We want to point to a substantial difference between, on one hand, the bundle Schrödinger equation (2.27), or (2.36), or (2.30) and, on the other hand, its initial conditions (see (2.31) or (2.28)) or the conventional Schrödinger equation (I.2.6) and its initial conditions (I.2.7). The bundle Schrödinger equations are absolutely invariant in a sense that they do not depend on any coordinates, space (-time) points or reference paths such as γ and hence, in our interpretation, are observer independent. In this vein, the bundle Schrödinger equations are analogous to the covariant equations in general relativity, which, due to their tensorial character, have similar properties. In contrast to the mentioned observation, the initial bundle conditions depend on the reference path γ, i.e. are observer dependent as, for example, the conventional Hamiltonian $\mathcal{H}$ is[11]. Consequently in the Hilbert bundle description the observer dependence, i.e. the dependence on the reference path γ, is 'moved' from the equations of motion to their initial conditions. It is clear that this dependence cannot be removed completely due to the equivalence between the Hilbert space and Hilbert bundle descriptions of quantum mechanics.

Since now we have at our disposal the machinery required for analysis of [14], we, as promised in section I.1, want to make some comments on it. In [14, p. 1455, left column, paragraph 4] it is stated 'that in the Heisenberg gauge (picture) the Hamiltonian is the null operator'. If so, all eigenvalues of the Hamiltonian vanish and, as they are picture independent, they are null in any picture of quantum mechanics. Consequently, from here one deduces the absurd conclusion that the 'energy levels of any system coincide and correspond to one and the same energy equal to zero'. Since the paper [14] is mathematically completely correct and rigorous, there is something wrong with the physical interpretation of the mathematical scheme developed in it. Without going into details, we describe below the solution of this puzzle, which simultaneously throws a bridge between [14] and the present investigation.

In [14] the system's Hilbert space $\mathfrak{H}$ is replace by a differentiable Hilbert bundle $\mathfrak{E}(\mathbb{R}_+, \mathfrak{H})$ (in our terms $(\mathfrak{E}, \pi, \mathbb{R}_+)$ with a fibre $\mathfrak{H}$), $\mathbb{R}_+ := \{t : t \in \mathbb{R}, t \geqslant 0\}$, which is an associated Hilbert bundle of the principle fibre bundle $\mathfrak{P}\big(\mathbb{R}_+, \mathfrak{U}(\mathfrak{H})\big)$ of orthonormal bases of $\mathfrak{H}$ where $\mathfrak{U}(\mathfrak{H})$ is the unitary group of (linear) bounded invertible operators in $\mathfrak{H}$ with bounded inverse. Let $p : \mathfrak{U}(\mathfrak{H}) \to GL(\dim \mathfrak{H}, \mathbb{C})$ be a (linear and continuous) representation of $\mathfrak{U}(\mathfrak{H})$ in the general linear group of $\dim \mathfrak{H}$-dimensional matrices. An obvious observation is that [14, equation (4.6)] under p transforms, up to notation, to our equation (2.24) (in [14] $\hbar = 1$ is taken). Thus, we

[11] For instance, suppose two point-like free particles 1 and 2 have masses m_a and momentum operators p_a, $a = 1, 2$, with respect to some observer. The particle's Hamiltonians are $\mathcal{H}_a = p_a^2/2m$, $a = 1, 2$. The Hamiltonian of the second particle with respect to the first one is $\mathcal{H}_{1,2} = p^2/2m$ (after the elimination of the centre of mass movement) with $p := (m_2 p_1 - m_1 p_2)/m_1 m_2$ and $m := m_1 m_2/(m_1 + m_2)$. For details, see [13, chapter 9, sections 11, 12].

see that what in [14] is called the Hamiltonian is actually the (analogue of the) matrix-bundle Hamiltonian $\boldsymbol{H}^{m}_{\gamma}(t)$, not the Hamiltonian $\mathcal{H}$ itself. This immediately removes the above-mentioned conflict: as we shall see later in the third part of this series, along any γ (or, over $\mathbb{R}_{+}$ in the notation of [14]—see below), we can choose a field of frames (bases) in which $\boldsymbol{H}^{m}_{\gamma}(t)$ identically vanishes but, due to (2.13), this does not imply the vanishing of the Hamiltonian at all. This particular choice of the frame along γ corresponds to the 'Heisenberg gauge' in [14], normally known as the Heisenberg picture.

Having in mind the above, we can describe [14] as follows. In it we have $F = \mathfrak{C}$, $M = \mathbb{R}_{+}$, $\mathcal{F} = \mathfrak{H}$ (the conventional system's Hilbert space), $J = \mathbb{R}_{+}$, $\gamma = \mathsf{id}_{\mathbb{R}_{+}}$ (other choices of γ correspond to reparametrization of the time) and $\frac{\partial}{\partial t}$, $t \in \mathbb{R}_{+}$ is the analogue of D^{+} in [14]. As we have already pointed out, the matrix-bundle Hamiltonian $\boldsymbol{H}^{m}_{\gamma}(t)$ represents the operator $A(t)$ of [14], incorrectly identified there with the 'Hamiltonian', and the choice of a field of bases over $\gamma(J) = \mathbb{R}_{+} = M$ corresponds to an appropriate 'choice of the gauge' in [14]. Now, after a correspondence between [14] and this paper is set, one can see that under the representation p the main results of [14], expressed by [14, equations (4.5), (4.6) and (4.8)], correspond to our equations (2.25) (see also (2.20)), (2.24) and (2.5) respectively.

Ending the comment on [14], we note two things. First, this paper uses a rigorous mathematical base, analogous to that in [15], which is not a goal of our work. Second, the ideas of [14] are a very good motivation for the present investigation and are helpful for its better understanding.

3. The bundle description of observables

In quantum mechanics it is accepted that to any dynamical variable, say $\mathbb{A}$, there corresponds a unique observable, say $\mathcal{A}(t)$, which is a Hermitian linear operator in the Hilbert space $\mathcal{F}$, i.e. $\mathcal{A}(t) : \mathcal{F} \to \mathcal{F}$, $\mathcal{A}(t)$ is linear, and $\mathcal{A}^{\dagger} = \mathcal{A}$ [3, 15, 16].

The mean value of an observable $\mathcal{A}$ in a state with state vector $\psi \in \mathcal{F}$ with finite norm is calculated according to (I.2.11). It is interpreted as an observed (mean) value of the dynamical variable $\mathbb{A}$ at a state ψ. This assumption and the probabilistic interpretation of the wavefunction ψ are the main tools for predicting experimentally observable results in quantum mechanics. As we said earlier in section I.4.3, the latter of these tools is transferred in Hilbert bundle quantum mechanics in an evident way. The bundle version of the former is the main task of this section. Below it will be shown that the proper bundle analogue of $\mathcal{A}$ is a suitable lifting of paths (in the bundle of restricted morphisms of the Hilbert bundle of states) or a (generally multiple-valued) morphism along paths (in the system's Hilbert bundle).

3.1. Heuristic introduction

Let $\psi^{(\lambda)}(t) \in \mathcal{F}$ be an eigenvector of $\mathcal{A}(t)$ with eigenvalue λ ($\in \mathbb{R}$), i.e. $\mathcal{A}(t)\psi^{(\lambda)}(t) = \lambda\psi^{(\lambda)}(t)$. According to (I.4.3) $\psi^{(\lambda)}(t)$ corresponds to the vector $\Psi^{(\lambda)}_{\gamma}(t) = l^{-1}_{\gamma(t)}\psi^{(\lambda)}(t) \in F_{\gamma(t)}$ in the bundle description. However, the Hilbert space and Hilbert bundle descriptions of a quantum evolution should be fully equivalent. Hence to $\mathcal{A}(t)$ in $F_{\gamma(t)}$ there should correspond a certain operator, which we denote by $A_{\gamma}(t)$. We define this operator by demanding every $\Psi^{(\lambda)}_{\gamma}(t)$ to be its eigenvector with eigenvalue λ, i.e. $\left(A_{\gamma}(t)\right)\Psi^{(\lambda)}_{\gamma}(t) := \lambda\Psi^{(\lambda)}_{\gamma}(t)$. Combining this equality with the preceding two, we easily verify that $\left(A_{\gamma}(t) \circ l^{-1}_{\gamma(t)}\right)\psi^{(\lambda)}(t) = \left(l^{-1}_{\gamma(t)} \circ \mathcal{A}(t)\right)\psi^{(\lambda)}(t)$, where the linearity of l_x has been used. Admitting that $\{\psi^{(\lambda)}(t)\}$ is a complete set of vectors, i.e. a basis of $\mathcal{F}$, we find

$$A_{\gamma}(t) = l^{-1}_{\gamma(t)} \circ \mathcal{A}(t) \circ l_{\gamma(t)} : F_{\gamma(t)} \to F_{\gamma(t)}. \tag{3.1}$$

More 'physically', the same result is derivable from (I.2.11) too. The mean value $\langle \mathcal{A}\rangle^t_\psi$ of $\mathcal{A}$ at a state $\psi(t)$ is given by (I.2.11) and the mean value of $A_\gamma(t)$ at a state $\Psi_\gamma(t)$ is

$$\langle A_\gamma(t)\rangle_{\Psi_\gamma(t)} := \langle A_\gamma(t)\rangle^t_{\Psi_\gamma} := \frac{\langle \Psi_\gamma(t)|A_\gamma(t)\Psi_\gamma(t)\rangle_{\gamma(t)}}{\langle \Psi_\gamma(t)|\Psi_\gamma(t)\rangle_{\gamma(t)}} \tag{3.2}$$

i.e. is it is given via (I.2.11) in which the scalar product $\langle\cdot|\cdot\rangle_x$, defined by (I.3.1), is used instead of $\langle\cdot|\cdot\rangle$. We define $A_\gamma(t)$ by demanding

$$\langle \mathcal{A}(t)\rangle^t_\psi = \langle A_\gamma(t)\rangle^t_{\Psi_\gamma}. \tag{3.3}$$

Physically this condition is quite natural as it means that the observed values of the dynamical variables are independent of the way we calculate them. From this equality, (I.4.1) and (I.3.1), we obtain $\langle\psi(t)|\,\mathcal{A}(t)\psi(t)\rangle = \langle\psi(t)|l_{\gamma(t)} \circ A_\gamma(t) \circ l^{-1}_{\gamma(t)}\psi(t)\rangle$, which, again, leads to (3.1). Thus we have also proved the equivalence of (3.1) and (3.3).

The above considerations lead to the idea that to every observable $\mathcal{A}$ at a moment t there should correspond an operator $A_\gamma(t)$, given by (3.1), in the fibre $F_{\gamma(t)} = \pi^{-1}(\gamma(t))$. It is almost evident, if $\gamma : J \to M$ is without self-intersections, that the collection of maps $\{A_\gamma(t)|t \in J\}$ forms a morphism over $\gamma(J)$ of the Hilbert bundle of the system restricted on $\gamma(J)$.

3.2. Rigorous considerations

As mentioned earlier (see section I.4), postulates I.4.1 and I.4.2 are not enough for the bundle description of observables. The contents of section 3.1 confirm this opinion. Relying on the above not quite rigorous results, we formulate the missing section of the chain as the next postulate.

Postulate 3.1. *Let (F, π, M) be the Hilbert bundle of a quantum system, $\gamma : J \to M$, and $t \in J$. In the bundle description of quantum mechanics, every dynamical variable $\mathbb{A}$ characterizing the system is represented by a map A assigning to the pair (γ, t) a map $A_\gamma(t) : \pi^{-1}(\gamma(t)) \to \pi^{-1}(\gamma(t))$ such that*

$$A_\gamma(t) = l^{-1}_{\gamma(t)} \circ \mathcal{A}(t) \circ l_{\gamma(t)} \tag{3.4}$$

where $\mathcal{A}(t) : \mathcal{F} \to \mathcal{F}$ is the linear Hermitian operator (in the system's Hilbert space $\mathcal{F}$) representing $\mathbb{A}$ in the conventional quantum mechanics. If at a moment $t \in J$ the system is in a state characterized by a bundle state vector $\Psi_\gamma(t)$ with a finite norm (in $F_{\gamma(t)}$), the observed value of $\mathbb{A}$ (or of A) with respect to γ at a moment t is equal to the mean value of $A_\gamma(t)$ in $\Psi_\gamma(t)$, which, by definition, is given by (3.2).

From (3.4), (3.2), (I.4.3) and (I.2.11), we derive (3.3). This simple result has a fundamental meaning: the observed values of a dynamical variable are (and must be!) independent of the way they are calculated. This assertion may be called the '*principle of invariance of the observed (mean) values*' and its essence is the independence of the physically measurable quantities of the mathematical way we describe them. In our context, it means the coincidence of the observed values of a dynamical variable calculated in the Hilbert bundle and Hilbert space descriptions. In other words, we can express the same by saying that the predictions of both conventional and bundle versions of quantum mechanics are absolutely identical regardless of the existence of three free parameters (the base M, the set $\{l_x|x \in M\}$ and the path γ) in the bundle case.

Let us now clarify the mathematical nature of the mapping A introduced via postulate 3.1. First of all, the maps $A_\gamma(t)$ are linear as $\mathcal{A}$ and $l_{\gamma(t)}$ are (see (3.4)). If we define A as a map $A : \gamma \mapsto A_\gamma$ with $A_\gamma : t \mapsto A_\gamma(t)$, we see that $A_\gamma : J \to F^M_0$, where

$$F^M_0 := \{\varphi_x \,|\, \varphi_x : F_x \to F_x,\ x \in M\} = \{\varphi_x \,|\, \varphi_x = \varphi|_{F_x},\ x \in M,\ \varphi \in \mathrm{Mor}_M(F, \pi, M)\}$$

is the bundle space of the bundle $\mathrm{mor}_M(F, \pi, M)$ of restricted morphisms over M (see section I.4.1). Since the bundles (F, π, M) and $\mathrm{mor}_M(F, \pi, M)$ have a common base, the manifold M, we conclude that A_γ is a lifting of $\gamma : J \to M$ in $\mathrm{mor}_M(F, \pi, M)$ (not in (F, π, M)!). Consequently, the map A, as considered above, is a *lifting of paths in the bundle of restricted M-morphisms of the system's Hilbert bundle of states*,

$$A \in \mathrm{PLift}\big(\mathrm{mor}_M(F, \pi, M)\big). \tag{3.5}$$

The linear maps $A_\gamma(t) : F_{\gamma(t)} \to F_{\gamma(t)}$ are Hermitian. Indeed, using (I.3.7) and (I.3.8) for $y = x = \gamma(t)$ and $A_{x\to x} = A_\gamma(t)$, and (3.4), we obtain

$$A^\ddagger_\gamma(t) = A_\gamma(t) \tag{3.6}$$

where the Hermiticity of $\mathcal{A}$ was used. A lifting A_γ in $\mathrm{mor}_M((F, \pi, M))$ of $\gamma : J \to M$ is called Hermitian if (3.6) holds for every $t \in J$; we denote this symbolically by writing $A^\ddagger_\gamma = A_\gamma$. Respectively, a lifting A in $\mathrm{PLift}(\mathrm{mor}_M(F, \pi, M))$ is Hermitian, $A^\ddagger = A$, if $A : \gamma \mapsto A_\gamma$ and $A^\ddagger_\gamma = A_\gamma$ for every path $\gamma \in \mathrm{P}(M)$ in M.

Let us summarize. In the bundle description a dynamical variable $\mathbb{A}$ is represented by a Hermitian lifting A of paths in the bundle of restricted morphisms over the base of the Hilbert bundle of states. For A equations (3.4) hold and its mean value along γ at a moment t for a system with state lifting Ψ is

$$\langle A\rangle^{t,\gamma}_\Psi := \langle A_\gamma(t)\rangle^t_{\Psi_\gamma} \tag{3.7}$$

with the rhs of this equality given by (3.2).

The map A, provided via postulate 3.1, can also be considered as a (multiple-valued) morphism along paths of the Hilbert bundle of states[12]. On one hand, define $A : \gamma \mapsto {}_\gamma A$ with ${}_\gamma A : x \mapsto \{A_\gamma(t) | \gamma(t) = x,\ t \in J\}$ for $x \in \gamma(J)$. If γ is without self-intersections, then ${}_\gamma A$ is in $\mathrm{Mor}_{\gamma(J)}(F, \pi, M)|_{\gamma(J)}$ (see section I.4.1). On the other hand, we can define $A : \gamma \mapsto {}_\gamma A$ by ${}_\gamma A : \pi^{-1}(\gamma(J)) \to \pi^{-1}(\gamma(J))$ with ${}_\gamma A|_{\pi^{-1}(x)} = \{A_\gamma(t) | \gamma(t) = x,\ t \in J\}$. In this case, if γ is without self-intersections, ${}_\gamma A \in \mathrm{Mor}_{\gamma(J)}(F, \pi, M)|_{\gamma(J)}$, i.e. up to a bijective map ${}_\gamma A$ is in $\mathrm{Sec}\big(\mathrm{mor}_{\gamma(J)}(F, \pi, M)|_{\gamma(J)}\big)$. Recalling that a morphism φ over M along paths of a bundle (E, π, B) is a map $\varphi : \gamma \mapsto \varphi_\gamma \in \mathrm{Mor}_{\gamma(J)}(F, \pi, M)|_{\gamma(J)}$ for every path $\gamma \in \mathrm{P}(B)$, we see that A is a morphism over M along paths without self-intersections. However, if γ is not injective, the map $A : \gamma \mapsto {}_\gamma A$ is, generally, a multiple-valued morphism (over M) along paths of (F, π, M) and it gives an alternative description of the map A introduced via postulate 3.1. If the multiplicity of A as a morphism along paths is really presented, this description will rarely be employed; if A as a morphism is single valued, it is somewhat 'simpler' to consider A as a morphism than as a lifting of paths and, therefore, this interpretation will be preferred.

Definition 3.1. *The unique Hermitian lifting of paths in the bundle of restricted morphisms (over the base of the Hilbert bundle of states) corresponding to a dynamical variable will be called an observable lifting (of paths); the corresponding (multiple-valued) morphism along paths of the Hilbert bundle of states will be called an observable morphism (along paths).*

By virtue of (3.6), the observable morphisms along paths are Hermitian,

$$A^\ddagger = A \tag{3.8}$$

which is also a corollary of (3.4) and (I.3.15).

Generally, to every operator $\mathcal{A} : \mathcal{F} \to \mathcal{F}$ there corresponds a unique (global) morphism $\overline{A} \in \mathrm{Mor}(F, \pi, M)$ given by

$$\overline{A}_x = \overline{A}\big|_{F_x} = l_x^{-1} \circ \mathcal{A} \circ l_x \qquad x \in M \qquad \mathcal{A} : \mathcal{F} \to \mathcal{F}. \tag{3.9}$$

[12] Cf the analogous situation concerning state liftings and sections in section I.4.3.

Consequently to an observable $\mathcal{A}(t)$ can be assigned the (global) morphism $\overline{A}(t)$, $\overline{A}(t)|_{F_x} = l_x^{-1} \circ \mathcal{A}(t) \circ l_x$, but this morphism $\overline{A}(t)$ is almost useless for our goals as it simply gives in any fibre F_x a linearly isomorphic image of the initial observable $\mathcal{A}(t)$ (see section I.4).

Notice that $A_\gamma(t)$ generally depends explicitly on t even if $\mathcal{A}$ does not. In fact, from (3.1) we obtain

$$\frac{\partial \boldsymbol{A}_\gamma(t)}{\partial t} = \left[\boldsymbol{g}_\gamma(t), \boldsymbol{A}_\gamma(t)\right]_- + \boldsymbol{l}_{\gamma(t)}^{-1}(t)\frac{\partial \boldsymbol{\mathcal{A}}(t)}{\partial t}\boldsymbol{l}_{\gamma(t)}(t) \tag{3.10}$$

where $[\cdot,\cdot]_-$ denotes the commutator of corresponding quantities, and

$$\boldsymbol{g}_\gamma(t) := -\boldsymbol{l}_{\gamma(t)}^{-1}(t)\frac{\mathrm{d}\boldsymbol{l}_{\gamma(t)}(t)}{\mathrm{d}t}. \tag{3.11}$$

In particular, to the Hamiltonian $\mathcal{H}$ in $\mathcal{F}$ there corresponds the *bundle Hamiltonian H* given by

$$H_\gamma(t) := l_{\gamma(t)}^{-1} \circ \mathcal{H}(t) \circ l_{\gamma(t)}. \tag{3.12}$$

This is an observable lifting of paths or morphism along paths.

From (3.12), using (I.2.9) and (I.5.10), we find

$$H_\gamma(t) = \mathrm{i}\hbar l_{\gamma(t)}^{-1} \circ \frac{\partial\, \mathcal{U}(t,t_0)}{\partial t} \circ l_{\gamma(t_0)} \circ U_\gamma(t_0,t). \tag{3.13}$$

From here a relationship between the matrix-bundle Hamiltonian and the bundle Hamiltonian can be obtained. For this purpose, we write (3.13) in a matrix form and, using (2.18) and $\mathrm{d}f_a(t)/\mathrm{d}t = E_a^{\ b} f_b(t)$, we obtain

$$\boldsymbol{H}_\gamma(t) = \boldsymbol{H}_\gamma^m(t) + \mathrm{i}\hbar\boldsymbol{l}_{\gamma(t)}^{-1}(t)\left(\frac{\mathrm{d}\boldsymbol{l}_{\gamma(t)}(t)}{\mathrm{d}t} + \boldsymbol{E}(t)\boldsymbol{l}_{\gamma(t)}(t)\right). \tag{3.14}$$

Substituting here (2.13), we obtain

$$\boldsymbol{H}_\gamma(t) = \boldsymbol{l}_{\gamma(t)}^{-1}(t)\boldsymbol{\mathcal{H}}(t)\boldsymbol{l}_{\gamma(t)}(t) \tag{3.15}$$

which is simply the matrix form of (3.12). Combining (3.14) with (2.14), we find the following connection between the matrix of the bundle Hamiltonian and the matrix Hamiltonian:

$$\boldsymbol{H}_\gamma(t) = \boldsymbol{l}_{\gamma(t)}^{-1}(t)\boldsymbol{\mathcal{H}}^m(t)\boldsymbol{l}_{\gamma(t)}(t) + \mathrm{i}\hbar\boldsymbol{l}_{\gamma(t)}^{-1}(t)\boldsymbol{E}(t)\boldsymbol{l}_{\gamma(t)}(t). \tag{3.16}$$

Notice that, due to (3.9) as well as to (3.1), to the identity map of $\mathcal{F}$ there corresponds a morphism along paths equal to the identity map of F:

$$\mathsf{id}_{\mathcal{F}} \longleftrightarrow \mathsf{id}_F. \tag{3.17}$$

3.3. Functions of observables

The results expressed by (3.1) and (3.9) enable functions of observables in $\mathcal{F}$ to be transferred into functions of liftings of paths (morphisms along paths) or morphisms of (F,π,M), respectively. We will illustrate this in the case of, for example, two variables.

Let $\mathcal{G} : (\mathcal{A}(t), \mathcal{B}(t)) \mapsto \mathcal{G}(\mathcal{A}(t), \mathcal{B}(t)) : \mathcal{F} \to \mathcal{F}$ be a function of the observables $\mathcal{A}(t)$, $\mathcal{B}(t) : \mathcal{F} \to \mathcal{F}$. It is natural to define the bundle analogue G of $\mathcal{G}$ by

$$G : (A,B) \mapsto G(A,B) : \gamma \mapsto G_\gamma(A,B) : \pi^{-1}(\gamma(J)) \to \pi^{-1}(\gamma(J))$$

where $G_\gamma(A,B)$ is a lifting of γ and

$$\begin{aligned} G_\gamma(A,B)\big|_t : &= l_{\gamma(t)}^{-1} \circ \mathcal{G}(\mathcal{A}(t), \mathcal{B}(t)) \circ l_{\gamma(t)} \\ &= l_{\gamma(t)}^{-1} \circ \mathcal{G}(l_{\gamma(t)} \circ A_\gamma(t) \circ l_{\gamma(t)}^{-1}, l_{\gamma(t)} \circ B_\gamma(t) \circ l_{\gamma(t)}^{-1}) \circ l_{\gamma(t)}. \end{aligned} \tag{3.18}$$

Thus $G(A, B)$ is an observable lifting of paths. This definition becomes evident in the cases when $\mathcal{G}$ is a polynomial or if it is expressible as a convergent power series; in both cases the multiplication has to be understood as an operator composition. If we are dealing with one of these cases, the definition (3.18) follows from the fact that for any observable liftings $A_1, \ldots, A_k$, $k \in \mathbb{N}$, of paths, the equality

$$A_{1,\gamma}(t) \circ A_{2,\gamma}(t) \circ \cdots \circ A_{k,\gamma}(t) = l_{\gamma(t)}^{-1} \circ \big(\mathcal{A}_1(t) \circ \mathcal{A}_2(t) \circ \cdots \circ \mathcal{A}_k(t)\big) \circ l_{\gamma(t)} \tag{3.19}$$

holds due to (3.1). In these cases $G(A, B)$ depends only on A and B and it is explicitly independent of the isomorphisms l_x, $x \in M$.

The commutator of two operators is a an important operator function in quantum mechanics. In the Hilbert space and bundle descriptions it is defined, respectively, by $[\mathcal{A}, \mathcal{B}]_- := \mathcal{A} \circ \mathcal{B} - \mathcal{B} \circ \mathcal{A}$ and $[A, B]_- := A \circ B - B \circ A$, where (see (3.18)) $(A \circ B) : \gamma \mapsto (A \circ B)_\gamma : t \mapsto (A \circ B)_\gamma(t) = A_\gamma(t) \circ B_\gamma(t)$. The relation

$$[A_\gamma(t), B_\gamma(t)]_- = l_{\gamma(t)}^{-1} \circ [\mathcal{A}, \mathcal{B}]_- \circ l_{\gamma(t)} \tag{3.20}$$

is an almost evident corollary of (3.1). It can also be considered as a special case of (3.18). In particular, to commuting observables (in $\mathcal{F}$) there correspond commuting observable liftings or morphisms:

$$[\mathcal{A}, \mathcal{B}]_- = 0 \iff [A, B]_- = 0. \tag{3.21}$$

A little more general is the result, following from (3.20), that to observables whose commutator is a c-number there correspond observable liftings with the same c-number as a commutator:

$$[\mathcal{A}, \mathcal{B}]_- = c\,\mathsf{id}_{\mathcal{F}} \iff [A, B]_- = c\,\mathsf{id}_F \tag{3.22}$$

for some $c \in \mathbb{C}$. In particular, the bundle analogue of the famous relation $[\mathcal{Q}, \mathcal{P}]_- = \mathrm{i}\hbar\,\mathsf{id}_{\mathcal{F}}$ between a coordinate $\mathcal{Q}$ and its conjugate momentum $\mathcal{P}$ is $[Q, P]_- = \mathrm{i}\hbar\,\mathsf{id}_F$.

A little more complicated is the case for operators and liftings of paths at different 'moments'. Let $\gamma : J \to M$ and $r, s, t \in J$. If $\breve{\mathcal{G}}_{s,t} : (\mathcal{A}, \mathcal{B}) \mapsto \mathcal{G}(\mathcal{A}(s), \mathcal{B}(t))$, we define the bundle analogue $\breve{G}_{s,t}$ of $\breve{\mathcal{G}}_{s,t}$ by

$$\breve{G}_{s,t} : (A, B) \mapsto \breve{G}_{s,t}(A, B) : \gamma \mapsto \breve{G}_{\gamma;s,t}(A, B) : \pi^{-1}(\gamma(J)) \to \pi^{-1}(\gamma(J))$$

where

$$\begin{aligned} \breve{G}_{\gamma;s,t}(A, B)\Big|_r &:= l_{\gamma(r)}^{-1} \circ \mathcal{G}(\mathcal{A}(s), \mathcal{B}(t)) \circ l_{\gamma(r)} \\ &= l_{\gamma(r)}^{-1} \circ \mathcal{G}(l_{\gamma(r)} \circ \breve{\mathcal{A}}_{\gamma;s}(r) \circ l_{\gamma(r)}^{-1}, l_{\gamma(r)} \circ \breve{\mathcal{B}}_{\gamma;t}(r) \circ l_{\gamma(r)}^{-1}) \circ l_{\gamma(r)} : F_{\gamma(r)} \to F_{\gamma(r)}. \end{aligned} \tag{3.23}$$

Here

$$\breve{\mathcal{A}}_{\gamma;t}(r) := l_{\gamma(r)}^{-1} \circ \mathcal{A}(t) \circ l_{\gamma(r)} = l_{t\to r}^{\gamma} \circ A(t) \circ l_{r\to t}^{\gamma} : F_{\gamma(r)} \to F_{\gamma(r)} \tag{3.24}$$

where (3.1) has been used and $l_{s\to t}^{\gamma} := l_{\gamma(s)\to\gamma(t)}$ is the (flat) linear transport (along paths) from $\gamma(s)$ to $\gamma(t)$ assigned to the isomorphisms l_x, $x \in M$ (see equation (I.3.13))[13]. Now the analogue of (3.19) is

$$\begin{aligned} &\breve{\mathcal{A}}_{1;\gamma;t_1}(r) \circ \breve{\mathcal{A}}_{2;\gamma;t_2}(r) \circ \cdots \circ \breve{\mathcal{A}}_{k;\gamma;t_k}(r) \\ &\qquad = l_{\gamma(r)}^{-1} \circ (\mathcal{A}_1(t_1) \circ \mathcal{A}_2(t_2) \circ \cdots \circ \mathcal{A}_k(t_k)) \circ l_{\gamma(r)}. \end{aligned} \tag{3.25}$$

[13] According to [17, sections 2 and 3] the observable lifting $\breve{A}_{\gamma;t}(r)$ along γ is obtained via linear transportation of $A_\gamma(t)$ along γ by means of the linear transport induced by $l_{s\to t}^{\gamma}$ along paths in the bundle $\mathrm{mor}_M(F, \pi, M)$ of restricted morphisms over M of (F, π, M).

So, if $\mathcal{G}$ is a polynomial or a convergent power series, the observable lifting $\breve{G}_{\gamma;s,t}(A, B)$ along γ depends only on $\breve{A}_{\gamma;s}(r)$ and $\breve{B}_{\gamma;t}(r)$.

In particular for $\mathcal{G}(\cdot,\cdot) = [\cdot,\cdot]_-$, we have

$$\left[\breve{A}_{\gamma;s}(r), \breve{B}_{\gamma;t}(r)\right]_- = l^{-1}_{\gamma(r)} \circ [\ \mathcal{A}(s),\ \mathcal{B}(t)]_- \circ l_{\gamma(r)} \tag{3.26}$$

which for $s = r = t$ reduces to (3.20). In this case the analogues of (3.21) and (3.22) are

$$[\ \mathcal{A}(s),\ \mathcal{B}(t)]_- = 0 \quad \iff \left[\breve{A}_{\gamma;s}(r), \breve{B}_{\gamma;t}(r)\right]_- = 0 \tag{3.27}$$

$$[\ \mathcal{A}(s),\ \mathcal{B}(t)]_- = c\,\mathrm{id}_{\mathcal{F}} \iff \left[\breve{A}_{\gamma;s}(r), \breve{B}_{\gamma;t}(r)\right]_- = c\,\mathrm{id}_{F_{\gamma(r)}} \tag{3.28}$$

respectively.

The above considerations can *mutatis mutandis*, e.g. by replacing $\gamma(t)$ with x, $\mathcal{A}(t)$ with $\mathcal{A}$, A with $\overline{A}$ etc, be transferred to global morphisms of (F, π, M), but this is not needed for the present investigation.

4. Conclusion

Here we have continued to apply the fibre bundle formalism to nonrelativistic quantum mechanics. We derived different forms of the bundle Schrödinger equation which governs the time evolution of state liftings of paths in the Hilbert bundle description.

In the bundle description, as we have seen, the observables are described via Hermitian liftings of paths or morphisms along paths in suitable bundles. We also considered some technical problems connected with functions of observables.

In the future continuation of the present series we plan to consider from a fibre bundle point of view the following items: pictures and integrals of motion, mixed states, evolution transport curvature, interpretation of the theory and its possible further developments.

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